ON CERTAIN SPACE GENERALIZATIONS.

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By its essential concept, a rectangular figure of the nth dimension must be ultimately bounded by $2n$ figures of the next lower, $(n-1)$st dimension; must have 2^n vertices; and from every such vertex n mutually perpendicular lines must spring. On the other hand, each such line has two bounding points, and hence the figure must have $\frac{1}{2}n$ times as many edges as points, or a total of $n(2^{n-1})$.

In like manner, since from every edge starts a square of four edges in two dimensional space, from every edge two squares in three dimensions, three in four dimensions, and $n-1$ squares in n dimensions; it follows that the number of said squares in our n dimensional rectangle must be

$$\frac{(n)\ (n-1)}{2!}2^{n-2}.$$

By similar reasoning, the number of bounding cubes must be

$$\frac{(n)\,(n-1)\,(n-2)}{3!}2^{n-3}.$$

Hence, to sum up, the a-dimensional boundaries of any rectangular figure in n dimensions, must be $\frac{n!}{(n-a)!(a)!}.2^{n-a}$.

Similar reasoning shows that figures in n dimensions, analogous to the equilateral triangle, and the tetrahedron, will be bounded by $n+1$ figures of the next lower $(n-1)$st dimension; and have $n+1$ vertices, $\frac{(n+1)\,(n)}{2!}$ edges, $\frac{(n+1)\,(n)\,(n-1)}{3!}$ triangular faces, $\frac{(n+1)\,(n)\,(n-1)\,(n-2)}{4!}$ bounding tetrahedrons; and in brief, $\frac{(n+1)!}{(n-a)!\ (a+1)!}$ boundaries of the ath dimension.

From which it follows that the numerical values of the boundaries must rise and fall symmetrically to a medial value; the number $n+1$ of the vertices equalling that of the last $n-1$ boundaries, as stated; the edges and the $n-2$ boundaries each being $\frac{(n+1)\ (n)}{2!}$; the planes and the $n-3$ boundaries each being $\frac{(n+1)\,(n)\,(n-1)}{3!}$, and so on.

Note that the formulae hold true, even when the apparently absurd

question is asked concerning the number of higher dimensional boundaries in a lower dimensional figure, that is, when a is taken greater than n. For the negative and fractional, or reciprocal, result, which we then obtain, as for example, that there are $-\frac{1}{6}$ cubes in a square, means simply that, reversely and reciprocally, there are $+6$ squares in every cube.

Lastly, in extension of Euler's theorem that in all three-dimensional rectilinear solids the sum of their vertices and surfaces is equal to their edges plus 2, we can deduce this general formula, namely: In every even $(2n)$-dimensional rectilinear figure the algebraic sum of its boundaries, taken in their sequence alternately plus and minus, is always zero, while the similar summation of those of every odd $(2n+1)$-dimensional figure is always $+2$.

The proof of this is as follows: First, for rectangular and tetrahedroidal figures.

Any rectangular figure of the nth dimension can be generated by the rectangular movement in n dimensional space of the corresponding figure in the next lower $(n-1)$st dimension. Through this movement the vertices, edges, squares, and cubes, etc., of the generating figure must be given in duplicate; while each vertex must trace an extra edge, each edge a square, each square a cube, and so on. So that letting v, e, p, s, etc., stand for the vertices, edges, planes, surfaces, etc., of our generating $(n-1)$st dimensional figure; and v_1, e_1, p_1, s_1, etc., for those of the generated n dimensional figure, we have always $v_1=2v$; $e_1=2e+v$; $p_1=2p+e$; $s_1=2s+p$; etc.

Now let us represent by l the number and character of the ultimate $(n-2)$ boundaries of our $(n-1)$ figure, which figure we will also call an m. When the last two boundaries of our generated (n) figure must plainly be

$$l_1=(2l+\ldots) \text{ and } m=(2m+l).$$

The algebraic sum of the boundaries of our m figure of $(n-1)$ dimensions, with alternate terms taken $+$ and $-$, will be $+v-e+p-s+\ldots\ \ldots\mp l$, where the sign of the last term l will be $-$ or $+$ according as m is an even or an odd dimensional figure.

Similarly then the sum of the boundaries, taken alternately $+$ and $-$, of our generated n figure will be $+v_1-e_1+p_1-s_1+\ldots\mp l_1\pm m_1=+2v-(2e+v)+(2p+e)-(2s+p)+\ldots\mp(2l+\ldots)\pm(2m+l)=+v-e+p-s+\ldots\mp l \pm 2m$, the sign of the last term $2m$ being again $+$, when the n figure is an odd dimension, but $-$, when it is of an even dimension.

Therefore in any two successive rectangular figures of the $(n-1)$st and nth dimension respectively, the algebraic sum, as above, of the successive boundaries of n, taken alternately $+$ and $-$, will be always greater than those of $(n-1)$, by $+2$, if $(n-1)$ be even and n odd. But if conversely, $(n-1)$ is odd and n even, then the said boundaries of n are always less by -2 than those of $(n-1)$.

Beginning now with the even dimensional square, $+v-e$ is zero, and

in the odd dimensional cube $+v-e+p$ is $+2$; which we know to be correct. Hence the hypothetical fourth, sixth, eighth, etc., dimensional rectangles must always have this summation of their boundaries equal to zero, while in the case of the fifth, seventh, ninth, etc., dimensional rectangles this sum is $+2$.

Taking up now the higher dimensional figures, analogous to the equilateral triangle and the regular tetrahedron, again let $+v-e+p-s+\ldots\mp l$ be the summation of the vertices, edges, planes, solids, etc., up to the limiting l boundary of such an $(n-1)$st dimensional figure, which we will, as before, call m, the sign of the final boundary being once more $-l$, when m is an even dimensional figure, but $+l$ when it is odd; while $+v_1-e_1+p_1-s_1+\ldots\mp l_1\pm m_1$ will be, as before, the summation of the boundaries of the next higher n figure, the signs being $-l_1+m_1$ when n is of an odd dimension, but $+l_1-m_1$ when n is even.

Then the law is $v_1=v+1$; $e_1=e+v$; $p_1=p+e$; $s_1=s+p$; $\ldots m_1=m+l$. Hence $+v_1-e_1+p_1-s_1+\ldots\mp l_1\pm m_1=+(v+1)-(e+v)+(p+e)-(s+p)+\ldots\mp(l+\ldots)\pm(m+l)=+1\pm m=+2$ or zero.

So that here also the summation of the boundaries, taken alternately $+$ and $-$, is always equal to zero, when n is an even and $(n-1)$ an odd dimensional figure, but is equal to $+2$ when n is odd and $(n-1)$ even.

This then fully proves the theorem for at least rectangular and tetrahedroidal figures of the nth dimension. And that it is equally valid for any rectilinear figure whatsoever in n dimensions can also be shown.

For alike in cuboid, tetrahedroid, and the analogue of the tetrakaidecagon in the fourth dimension, from any corner must stretch four edges, six planes and four solids. So that, truncating a corner in any one of the figures, we will add four new vertices and lose an old one; or will add $+3$ vertices in all. We will create six new edges and three new planes but no new solids. hence our new figure adds $+3v-6e+3p=0$ and the previous summation of the untruncated figure is not affected,

But a still more general proof, valid in all dimensions is the following: By the dimensional concept, from every one-dimensional line must stretch two plane faces and one solid and from each plane face a solid, in all three dimensional figures. In those of the fourth dimension, every edge must touch three plane faces and three solids. While every face must bound and be common to two of the bounding solids. And thus in any n-dimensional figure each line must touch $(n-1)$ plane faces. Each face must touch $(n-2)$ solids and every solid must have $(n-3)$ fourth dimensional boundaries and so on, until at last each $(n-2)$ boundary must touch $(n-[n-2])=+2$ of the $(n-1)$ boundaries of our n-dimensional figure.

Therefore taking, say, any fourth-dimensional figure, let us build it up step by step from its component l solids. And let us represent the vertices, edges and planes of our first solid by $+v_1-e_1+p_1$; those of the second by $+v_2-e_2+p_2$, and so on.

Take the first solid $+v_1-e_1+p_1=+2$, and join it to the second solid, so that an identical face in each coincides. Now were we dealing with three-dimensional space, we should evidently have lost two plane faces, each having $v=e$; the said faces passing into the interior of our new composite solid figure, which solid figure has yet $+v-e+p=+2$.

But we are concerned with four-dimensional space, wherein all the planes, etc., of our first solid remain unaltered, and where the second and added solid loses but *one* face, with its equal vertices and edges, by coalescing with the similar face in the first solid. Hence now $v_2-e_2+p_2=+3$.

Similarly, adding the third bounding solid, in place of four planes disappearing, as they would in three-dimensional space, we lose by coalescing but two, and thus $+v_3-e_3+p_3=+4$.

And so on, until we come to the last solid but one, $(l-1)$. When as before, $+v_{l-1}-e_{l-1}+p_{l-1}=+l$.

But upon adding the last l bounding solid, no new boundaries appear. All of its vertices, edges, and planes coalescing with those already existing in our built up figure, so that $+v_l-e_l+p_l=+l=$the solid boundaries, by definition. And hence $+v-e+p-s=0$, for any and all fourth-dimensional figures.

And the same would be true, were a fifth-dimensional figure given, with m bounding fourth-dimensional figures. For, taking the first of the said m figures, we have $+v_1-e_1+p_1-s_1=0$. Adding the second, and thereby losing but one solid, in place of two, we have $+v_2-e_2+p_2-s_2=-1$. The third figure, in like manner, gives us $+v_3-e_3+p_3-s_3=-2$. Until finally, $+v_{m-1}-e_{m-1}+p_{m-1}-s_{m-1}=-(m-2)=+2-m=+v_m-e_m+p_m-s_m$. Or $+v-e+p-s+m=+2$.

And quite similarly, for any dimension. The odd dimensions always adding $+2$ to the zero summation of the previous even figure, and conversely, the even always taking -2 from the $+2$ summation of the previous odd, as we similarly proved for rectangular and tetrahedroidal figures.

Lastly, were we to count the n-dimensional figure itself as its own boundary, and thus add -1 or $+1$ to our previous summation, according as n is odd or even, then we obtain the still more simple rule that any and all such summations always give $+1$, whatever may be the rectilinear figure.

In this case the binomial development of $(2-1)^n$ will give us both the sequence and the summation of the boundaries of a rectangular n-dimensional figure; while, in a similar way, the symmetrical sequence and summation of those of an n-dimensional tetrahedroidal figure can be represented by $-(1-1)^{n+1}$, omitting the first term of the binomial development.